Problem Set 5

Time Dependence in Quantum Mechanics

1. For a particle in a box energy eigenfunction, $\psi_n(x)$, do the following.
   (a) Write the time-dependent wavefunction at some arbitrary time, t, in terms of the time-independent energy eigenstate, $\psi_n(x)$.
   (b) What is the expectation value of the total energy operator, $\hat{H}$, for $\Psi_n(x,t)$ at some time t?
   (c) What is the expectation value of the position operator, $\hat{x}$, for $\Psi_n(x,t)$ at some time t?

2. For each of the superposition states,

$$\phi_1(x) = N_1\left(\psi_1(x) + \psi_2(x)\right)$$
$$\phi_2(x) = N_2\left(3\psi_1(x) - \psi_2(x)\right)$$
$$\phi_3(x) = N_3\left(2\psi_1(x) + \sqrt{2}\psi_2(x) + \sqrt{3}\psi_3(x)\right)$$
$$\phi_4(x) = N_4\left(\psi_1(x) + 0.7\psi_2(x) + 0.4\psi_3(x)\right)$$
$$\phi_5(x) = N_5\left(3\psi_1(x) + \psi_2(x) + 2\psi_3(x) + \sqrt{2}\psi_4(x)\right)$$

   do the following.
   (a) Write the time-dependent wavefunction, $\Phi(x,t)$ at some arbitrary time t, in terms of the time-independent energy eigenstates, $\psi_n(x)$.
   (b) Normalize the wavefunction at all times t.
   (c) What is the expectation value of the total energy operator, $\hat{H}$, for $\Phi(x,t)$ at some time t?
   (d) What is the expectation value of the position operator, $\hat{x}$, for $\Phi(x,t)$ at some time t?
   (e) What is the expectation value of the square of the position operator, $\hat{x}^2$, for $\Phi(x,t)$ at some time t?
   (f) What is the expectation value of the momentum operator, $\hat{p}_x$, for $\Phi(x,t)$ at some time t?
   (g) What is the expectation value of the square of the momentum operator, $\hat{p}_x^2$, for $\Phi(x,t)$ at some time t?

Some Potentially Useful Equations

Energy eigenfunctions:	$\psi_n(x) = \begin{cases} \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right),\ n = 1, 2, 3, \ldots & \text{if } 0 \leq x \leq L \\ 0 & \text{otherwise} \end{cases}$
Energy eigenvalues:	$E_n = \frac{n^2\pi^2\hbar^2}{2mL^2},\ n = 1, 2, 3, \ldots$
Time-dependence:	$\Psi_n(x,t) = \psi_n(x)\mathrm{e}^{-iE_n t/\hbar}$
Expectation value:	$\langle \hat{A} \rangle = \frac{\int_{-\infty}^{\infty} \Psi^*(x)\hat{A}\Psi(x)\mathrm{d}x}{\int_{-\infty}^{\infty} \Psi^*(x)\Psi(x)\mathrm{d}x}$
Position operator:	$\hat{x} = x$
Momentum operator:	$\hat{p}_x = -i\hbar\frac{\mathrm{d}}{\mathrm{d}x}$
Kinetic energy operator:	$\hat{T} = -\frac{\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}x^2}$
Potential energy operator:	$\hat{V} = V(x) = \begin{cases} 0 & \text{if } 0 \leq x \leq L \\ \infty & \text{otherwise} \end{cases}$
Hamiltonian operator:	$\hat{H} = \hat{T} + \hat{V}$
Kronecker delta:	$\delta_{mn} = \begin{cases} 1 & \text{if } m = n \\ 0 & \text{if } m \neq n \end{cases}$
Ket:	$\|n\rangle = \Psi_n(x)$
Bra:	$\langle n\| = \Psi_n^*(x)$
Bra-ket:	$\langle m\|\hat{A}\|n\rangle = \int_{-\infty}^{\infty} \Psi_m^*(x)\hat{A}\Psi_n(x)\mathrm{d}x$
Matrix form of an operator	$A_{mn} = \langle \Psi_m\|\hat{A}\|\Psi_n\rangle = \int_{-\infty}^{\infty} \Psi_m^*(x)\hat{A}\Psi_n(x)\mathrm{d}x$
Resolution of the identity	$\hat{I} = \sum_{n=1}^{\infty} \|\Psi_n\rangle\langle\Psi_n\|$
Commutator:	$[\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}$
Root-mean-square deviation:	$\Delta\hat{A} = \sqrt{\langle \hat{A}^2 \rangle - \langle \hat{A} \rangle^2}$
Heisenberg uncertainty principle	$\Delta A \Delta B \geq \frac{1}{2}\|\langle[\hat{A}, \hat{B}]\rangle\|$