Problem Set 7

Harmonic Oscillator

1. For the quantum harmonic oscillator, prove that $\psi_n(x)$ is an eigenfunction of the Hamiltonian for the lowest few quantum numbers ($n = 0, 1, 2, \ldots$). What is the corresponding eigenvalue?

2. Show that, $\Delta\hat{x}\Delta\hat{p}_x \geq \frac{\hbar}{2}$, using the general uncertainty relation, $\Delta\hat{A}\Delta\hat{B} \geq \frac{1}{2}|\langle[\hat{A},\hat{B}]\rangle|$, for the ground state harmonic oscillator wavefunction, $\psi_0(x)$.

3. Show that the lowest few harmonic oscillator energy eigenfunctions are orthonormal.

4. Determine the matrix elements of the position operator in a basis of the lowest few harmonic oscillator wavefunctions. $X_{ij} = \langle\psi_i|\hat{x}|\psi_j\rangle$

5. Determine the matrix elements of the square of the position operator in a basis of the lowest few harmonic oscillator wavefunctions. $Y_{ij} = \langle\psi_i|\hat{x}^2|\psi_j\rangle$

6. Determine the matrix elements of the momentum operator in a basis of the lowest few harmonic oscillator wavefunctions. $P_{ij} = \langle\psi_i|\hat{p}_x|\psi_j\rangle$

7. Determine the matrix elements of the square of the momentum operator in a basis of the lowest few harmonic oscillator wavefunctions. $Q_{ij} = \langle\psi_i|\hat{p}_x^2|\psi_j\rangle$

8. Determine the matrix elements of the Hamiltonian (total energy) operator in a basis of the lowest few harmonic oscillator wavefunctions. $H_{ij} = \langle\psi_i|\hat{H}|\psi_j\rangle$

Some Potentially Useful Equations

Energy eigenfunctions:	$\psi_n = A_n H_n(a^{1/2}x)\mathrm{e}^{-ax^2/2}, \;\; n = 0, 1, 2\ldots$
	$a = \sqrt{\frac{km}{\hbar^2}} \qquad A_n = \frac{1}{\sqrt{2^n n!}}\left(\frac{a}{\pi}\right)^{1/4}$
Energy eigenvalues:	$E_n = (n + \frac{1}{2})\hbar\sqrt{\frac{k}{m}}$
Expectation value:	$\langle \hat{A} \rangle = \frac{\int_{-\infty}^{\infty} \psi^*(x)\hat{A}\psi(x)\mathrm{d}x}{\int_{-\infty}^{\infty} \psi^*(x)\psi(x)\mathrm{d}x}$
Position operator:	$\hat{x} = x$
Momentum operator:	$\hat{p}_x = -i\hbar\frac{\mathrm{d}}{\mathrm{d}x}$
Kinetic energy operator:	$\hat{T} = -\frac{\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}x^2}$
Potential energy operator:	$\hat{V} = V(x) = \frac{1}{2}kx^2$
Hamiltonian operator:	$\hat{H} = \hat{T} + \hat{V}$
Matrix form of an operator	$A_{mn} = \langle \Psi_m \vert \hat{A} \vert \Psi_n \rangle = \int_{-\infty}^{\infty} \Psi_m^*(x)\hat{A}\Psi_n(x)\mathrm{d}x$
Commutator:	$[\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}$
Root-mean-square deviation:	$\Delta\hat{A} = \sqrt{\langle \hat{A}^2 \rangle - \langle \hat{A} \rangle^2}$
Heisenberg uncertainty principle	$\Delta A \Delta B \geq \frac{1}{2}\vert\langle[\hat{A}, \hat{B}]\rangle\vert$

Hermite polynomials

$H_0(z) = 1$

$H_1(z) = 2z$

$H_2(z) = 4z^2 - 2$

$H_3(z) = 8z^3 - 12z$

$H_4(z) = 16z^4 - 48z^2 + 12$

$H_5(z) = 32z^5 - 160z^3 + 120z$