Problem Set 8

Free Particle, Particle on a Ring, Particle on a Sphere, and Angular Momentum

1. The general solution to the free particle is one dimension is:

$$\psi(x) = A\mathrm{e}^{ikx} + B\mathrm{e}^{-ikx}.$$

   (a) Is $\psi(x)$ an eigenfunction of the momentum operator when $A = 1$ and $B = 0$? What is its eigenvalue?

   (b) Is $\psi(x)$ an eigenfunction of the momentum operator when $A = 0$ and $B = 1$? What is its eigenvalue?

2. An alternative form of $\psi(x)$ is

$$\psi(x) = C\cos(kx) + D\sin(kx).$$

   Show that these two forms of $\psi(x)$ are equivalent by writing the constants, C and D in terms of A and B. *Hint:* Use Euler's formula to expand the exponentials.

3. Show that the ψ_{m_l} wavefunctions are eigenfunctions of the particle on a ring Hamiltonian.

4. Show that the ψ_{m_l} wavefunctions are orthonormal.

5. Show that the ψ_{l,m_l} wavefunctions are eigenfunctions of the particle on a sphere Hamiltonian.

6. Show that the ψ_{l,m_l} wavefunctions are orthonormal.

7. Evaluate all pairs of commutators between $\hat{L}_x$, $\hat{L}_y$, $\hat{L}_z$, $\hat{L}^2$, $\hat{L}_+$ and $\hat{L}_-$. Here, $\hat{L}_+ = \hat{L}_x + i\hat{L}_y$ and $\hat{L}_- = \hat{L}_x - i\hat{L}_y$.

8. Rewrite $\hat{L}^2$ using $\hat{L}_+$ and $\hat{L}_-$ in place of $\hat{L}_x$ and $\hat{L}_y$.

9. Evaluate the expectation value of the $\hat{L}_z$ amd $\hat{L}_z^2$ angular momentum operators for the particle on a ring energy eigenfunctions.

10. Evaluate the expectation value of these angular momentum operators ($\hat{L}_x$, $\hat{L}_y$, $\hat{L}_z$, $\hat{L}^2$, $\hat{L}_+$ and $\hat{L}_-$) for particle on a sphere energy eigenfunctions, ψ_{l,m_l}.

Some Potentially Useful Equations

Angular momentum operators

$\hat{L}_x = -i\hbar\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right) = -i\hbar\left(-\sin\phi\frac{\partial}{\partial\theta} - \cot\theta\cos\phi\frac{\partial}{\partial\phi}\right)$

$\hat{L}_y = -i\hbar\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right) = -i\hbar\left(\cos\phi\frac{\partial}{\partial\theta} - \cot\theta\sin\phi\frac{\partial}{\partial\phi}\right)$

$\hat{L}_z = -i\hbar\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right) = -i\hbar\left(\frac{\partial}{\partial\phi}\right)$

$\hat{L}^2 = \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2$

Polar Coordinates

$x = r\cos\phi$

$y = r\sin\phi$

$\int_{x=-\infty}^{\infty}\int_{y=-\infty}^{\infty} f(x,y)\,\mathrm{d}x\,\mathrm{d}y \equiv \int_{\phi=0}^{2\pi}\int_{r=0}^{\infty} f(r,\phi)\,r\,\mathrm{d}r\,\mathrm{d}\phi$

Spherical Polar Coordinates

$x = r\sin\theta\cos\phi$

$y = r\sin\theta\sin\phi$

$z = r\cos\theta$

$\int_{x=-\infty}^{\infty}\int_{y=-\infty}^{\infty}\int_{z=-\infty}^{\infty} f(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \equiv \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\int_{r=0}^{\infty} f(r,\theta,\phi)\sin\theta\,r^2\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi$

Free Particle

$\hat{H} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}$

$\psi_k(x) = e^{ikx}, \quad k = -\infty\ldots\infty$

Particle on a Ring

$\hat{H} = -\frac{\hbar^2}{2mR^2}\frac{\partial^2}{\partial\phi^2}$

$\psi_{m_l}(\phi) = \frac{1}{\sqrt{2\pi}}e^{im_l\phi}, \;\; m_l = 0, \pm1, \pm2, \ldots$

$E_{m_l} = \frac{m_l^2\hbar^2}{2mR^2}$

Particle on a Sphere

$\hat{H} = -\frac{\hbar^2}{2mR^2}\Lambda^2$

$\Lambda^2 = \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2} + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta}$

$\psi_{l,m_l}(\theta,\phi) = Y_{l,m_l}(\theta,\phi), \;\; l = 0, 1, 2\ldots \;\; m_l = -l, \ldots, 0, \ldots, l$

$E_{l,m_l} = l(l+1)\frac{\hbar^2}{2mR^2}$

$Y_{0,0} = \frac{1}{2}\pi^{-1/2}$

$Y_{1,0} = \left(\frac{3}{4\pi}\right)^{1/2}\cos\theta$

$Y_{1,\pm1} = \mp\left(\frac{3}{8\pi}\right)^{1/2}\sin\theta\,\mathrm{e}^{\pm i\phi}$

$Y_{2,0} = \left(\frac{5}{16\pi}\right)^{1/2}(3\cos^2\theta - 1)$

$Y_{2,\pm1} = \mp\left(\frac{15}{8\pi}\right)^{1/2}\cos\theta\sin\theta\,\mathrm{e}^{\pm i\phi}$

$Y_{2,\pm2} = \left(\frac{15}{32\pi}\right)^{1/2}\sin^2\theta\,\mathrm{e}^{\pm 2i\phi}$